Wavelet denoising

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TSIA-SD205
Outline

Introduction
- Wavelet transform
- Principles of denoising

Denoising
- Oracles
- Minimax and Universal threshold
- SURE
- Bayes
Outline

Introduction
Wavelet transform
Principles of denoising

Denoising
Filterbanks 1D

Characteristics of wavelet transform

- Energy concentration
- Contours representation
- Multiple resolution analysis
  - Low resolution version
  - “Details”

\[ s[k], a[k], c[k] \]
Filterbanks 1D

Decomposition

\[
s[k] \xrightarrow{h} \hat{a}[k] \xrightarrow{\downarrow 2} a[k]
\]

\[
s[k] \xrightarrow{g} \hat{c}[k] \xrightarrow{\downarrow 2} c[k]
\]

Analysis Filterbank

\[
\downarrow 2: \text{decimation: } a[k] = \hat{a}[2k]
\]
Reconstruction

\[ a[k] \rightarrow \uparrow 2 \rightarrow \hat{a}[k] \rightarrow \tilde{h} \rightarrow \tilde{s}[k] \]

\[ c[k] \rightarrow \uparrow 2 \rightarrow \hat{c}[k] \rightarrow \tilde{g} \]

Synthesis Filterbank

\[ \uparrow 2: \text{interpolator, doubles the number of samples} \]

\[ \hat{a}[k] = \begin{cases} 
    a[k/2] & \text{if } k \text{ is even} \\
    0 & \text{if } k \text{ is odd}
\end{cases} \]
Properties of the filters

- Perfect reconstruction
  - It is possible to reconstruct the signal from its coefficients
- Finite Impulse Response
  - Finite operation implementation
- Orthogonality
  - Coefficients’ energy equal to signal energy
- Vanishing moments
  - Capacity of reproducing polynomial signals with zero details
- Symmetry
  - Implementation by symmetrization and periodization
Multiresolution Analysis 1D

Structure of wavelets decomposition with 3 levels of resolution
Multiresolution Analysis 1D
Multiresolution synthesis 1D

\[ a_0[k] = s[k] \]

Reconstruction from the wavelet coefficients
**2D MRA**

2D separable filterbanks

1 level of decomposition

\[
a_{j-1}[n, m] 
\]

\[
h[n] \quad (2, 1) \downarrow \quad h[m] \quad (1, 2) \downarrow \quad a_j[n, m]
\]

\[
g[n] \quad (2, 1) \downarrow \quad g[m] \quad (1, 2) \downarrow \quad c_j^H[n, m]
\]

\[
h[m] \quad (1, 2) \downarrow \quad c_j^V[n, m]
\]

\[
g[m] \quad (1, 2) \downarrow \quad c_j^D[n, m]
\]
Example
Example
(A), (H), (V) and (D) areas correspond to the approximation subband, the horizontal, vertical and diagonal detail subbands.
Introduction

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Principles of denoising

2D MRA

2D separable MRA with 3 resolution levels
AMR 2D - frequency interpretation

$\begin{align*}
&f_{\text{ver}} \\
&f_{\text{hor}}
\end{align*}$
Example
Example
Example
Example
Example
Example
Principles

Model: Observation: \( r(t) \); sum of an unknown signal \( s(t) \) and random noise \( b(t) \).
After decomposition on a wavelet basis:

\[
c^r_j[k] = c^s_j[k] + c^b_j[k]
\]

Hypothesis:

- Orthonormal basis, periodical decomposition
- Original signal (resolution \( j = 0 \)) has size multiple of \( 2^{j_{\text{max}}} \)
- High MSE in the approximation subband: \( a^s_{j_{\text{max}}} \approx a^r_{j_{\text{max}}} \)
Denoising

Principles

- \( c_j^r[k] = c_j^s[k] + c_j^b[k] \)
- High MSE in the approximation subband: \( a_{j_{\text{max}}}^s \approx a_{j_{\text{max}}}^r \)
- Estimator: \( \hat{s} \)
- Criterion: MSE minimization: \( \mathcal{E}^2(s) = E\{\|s - \hat{s}\|^2\} \)
- Without denoising, \( \hat{s} = r \)

\[
\begin{align*}
\mathcal{E}^2(s) &= E\{\|s - r\|^2\} = E\{\|c^s - c^r\|^2\} = E\{\|c^b\|^2\} = K_m \sigma^2 \\
K_m &= (1 - 2^{j_{\text{max}}})K
\end{align*}
\]
Denoising

Principles

- Regular signal
  - Energy concentrated in low frequencies
  - Sparse signal in high frequencies
  - Many very small coefficients
  - A few large coefficients (information!)
- Noise is often white and stationary
  - Model: white, stationary, centered and with power $\sigma^2$
  - Noise power is equally shared among subbands
- What do we find in the subbands?
Introduction
Denoising

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Principles of denoising

Examples

SNR: 22.4 dB;

\[ \sigma = 10 \]
Examples

Approximation subband

SNR: 46.4 dB
Examples

Detail subband

SNR: 15.2 dB
Noise variance estimation

Hypothesis:

- In the highest resolution subband the coefficients come only from the noise:

\[ \{ c_1^s[k] \}_{0 \leq k < K/2} \approx 0 \]

- Zero-mean Gaussian noise, i.i.d.
Noise variance estimation

We consider the distribution of $|Z|$, when $Z$ is normal.

We know that the median of $|Z|$ is $0.6745 \sigma$.

then we choose:

$$\hat{\sigma} = \frac{1}{0.6745} \text{Med}|c'_1|$$
Noise variance estimation

\[
\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}
\]

- Histogram
- \(\hat{\sigma}\)
- \(\sigma\)
Outline

Introduction

Denoising
- Oracles
- Minimax and Universal threshold
- SURE
- Bayes
Attenuation estimator

Definition:

\[ c_j^s[k] = \theta_j[k] c_j^r[k] \]

MSE:

\[
\mathcal{E}_a^2(s) = \sum_{j=1}^{j_{\max}} \sum_{k=0}^{K2^{-j}-1} \mathbb{E} \left[ \left( c_j^s[k] - \theta_j[k] c_j^r[k] \right)^2 \right]
\]

\[
(c^s - \theta c^r)^2 = (c^s(1 - \theta) - \theta c^b)^2
\]

\[
\mathcal{E}_a^2(s) = \sum_{j=1}^{j_{\max}} \sum_{k=0}^{K2^{-j}-1} (c^s)^2 (1 - \theta)^2 + \sigma^2 \theta^2
\]
Attenuation estimator

General term of the previous sum:

\[ J = (c_s^s)^2(1 - 2\theta + \theta^2) + \sigma^2\theta^2 \]

Minimizing wrt \( \theta \):

\[ \frac{\partial J}{\partial \theta} = -2(c_s^s)^2 + 2\theta((c_s^s)^2 + \sigma^2) \]

Then:

\[ \theta^* = \frac{(c_s^s)^2}{(c_s^s)^2 + \sigma^2} \]
Attenuation estimator

**Oracle**

\[
\theta^* = \frac{(c^s)^2}{(c^s)^2 + \sigma^2}
\]

**Oracle**: the estimator depends on the signal. Useful for theoretical bounds

\[
J = (c^s)^2 (1 - \theta)^2 + \sigma^2 \theta^2 = \frac{\sigma^2 (c^s)^2}{\sigma^2 + (c^s)^2}
\]

\[
\mathcal{E}_a^2(s) = \sum_{j=1}^{\text{max}} \sum_{k=0}^{K2^{-j} - 1} \frac{\sigma^2 (c_j^s[k])^2}{\sigma^2 + (c_j^s[k])^2}
\]
Attenuation estimator

Binary oracle

- If we constrain $\theta$ to be binary: $\theta_j[k] \in \{0, 1\}$, then
  - $J = (c^s)^2$ if $\theta = 0$; otherwise $J = \sigma^2$
  - then we choose $\theta = 0$ if $(c^s)^2 < \sigma^2$
- In this case, the MSE is:

$$E_0^2(s) = \sum_{j=1}^{j_{\text{max}}} \sum_{k=0}^{K2^{-j-1}} \min \left[ \sigma^2, (c_j^s[k])^2 \right]$$
Attenuation estimator

Binary oracle

\[
\mathcal{E}_o^2(s) = \sum_{j=1}^{j_{\text{max}}} \sum_{k=0}^{K2^{-j}-1} \min \left[ \sigma^2, (c_j^s[k])^2 \right]
\]

\[
\mathcal{E}_a^2(s) = \sum_{j=1}^{j_{\text{max}}} \sum_{k=0}^{K2^{-j}-1} \frac{\sigma^2(c_j^s[k])^2}{\sigma^2 + (c_j^s[k])^2}
\]

\[
0 < x \leq y \Rightarrow \frac{xy}{x+y} \geq \frac{xy}{2y} = \frac{1}{2}x = \frac{1}{2} \min(x, y)
\]

\[
\frac{1}{2} \mathcal{E}_o^2(s) \leq \mathcal{E}_a^2(s)
\]

\[
\mathcal{E}_a^2(s) \leq \mathcal{E}_o^2(s) \leq 2\mathcal{E}_a^2(s)
\]
Attenuation estimator

Binary oracle

- The MSE of the binary oracle is at most the double of the attenuation oracle
- Conclusion: We keep the wavelet coefficients where the signal can be supposed to be large, and we set to zero the others
- Simplified model: there are $Q$ non-zero coefficients $c^s$ and they are greater than $\sigma$; the others are zeros
- The binary oracle has in this case an MSE of $Q\sigma^2$
- Without denoising the MSE is $K_m\sigma^2$, where $K_m = K(1 - 2^{-j_{\text{max}}})$ is the number of available wavelet coefficients
Attenuation estimator

\[ s[k] \]

\[ a_1[k] \]

\[ c_1[k] \]

\[ a_2[k] \]

\[ c_2[k] \]

\[ c_1[k] \]

\[ a_4[k] \]

\[ c_4[k] \]

\[ c_3[k] \]

\[ c_2[k] \]

\[ c_1[k] \]
Attenuation estimator

\[ r[k] = s[k] + b[k] \]
Attenuation estimator

\[ a_4^\hat{s}[k], c_3^\hat{s}[k], c_2^\hat{s}[k], c_1^\hat{s}[k] \]
Binary oracle

- In conclusion, the binary oracle allows to reduce error by a factor:
  \[ \frac{K_m}{Q} \]
- A good wavelet basis is one making \( Q \) as small as possible
- The wavelet basis must generate a few large coefficients and many small ones
- In other words, it should concentrate energy in as few coefficients as possible
Thresholding

Hard Thresholding

\[ c^\hat{s} = \begin{cases} 
  c^r & \text{if } |c^r| > \chi \\
  0 & \text{if } |c^r| \leq \chi 
\end{cases} \]

Soft Thresholding

\[ c^\hat{s} = \begin{cases} 
  c^r - \chi & \text{if } c^r > \chi \\
  0 & \text{if } |c^r| \leq \chi \\
  c^r + \chi & \text{if } c^r < \chi 
\end{cases} \]
Thresholding

- Hard thresholding is not continuous near the threshold $\pm \chi$
- Soft thresholding introduces a bias $\mp \chi$ on the estimation of the large coefficients
- Main problem: value of the threshold $\chi$
  - *Minimax* method
  - *Visushrink* method (universal threshold)
  - *SURE* method
  - *Hybrid* method
Minimax method

Definitions:

- $K_m = K(1 - 2^{-j_{\text{max}}})$ is the number of available wavelet coefficients
- $\mathcal{E}_\chi$ is the MSE associated to the soft thresholding with threshold $\chi$
- $\tilde{\mu}$ is a symmetric probability density and $\bar{\mu}$ its normalized version (variance 1)

Hypothesis: The noise wavelets coefficients $c^b_j[k]$ have all the same symmetric PDF $\tilde{\mu}$ (of variance $\sigma^2$)

Then

- There is an equation giving $\chi_m$, the threshold minimizing the maximum MSE over signals $s$ for the soft thresholding case.
- This MSE can be related to the binary oracle case.
Minimax method

\[
\inf_{\chi \geq 0} \sup_s \frac{\mathcal{E}_\chi^2(s)}{\sigma^2 + \mathcal{E}_\sigma^2(s)} = \sup_s \frac{\mathcal{E}_{\chi m}^2(s)}{\sigma^2 + \mathcal{E}_\sigma^2(s)} = \Lambda_{\chi m} = \frac{K_m(\chi_m^2 + \sigma^2)}{(K_m + 1)\sigma^2},
\]

where \( \chi_m \) is the unique solution in \( \mathbb{R}_+ \) of the equation

\[
2(K_m + 1) \int_\chi^\infty (z - \chi)^2 \tilde{\mu}(z)dz = \chi^2 + \sigma^2.
\]
Minimax method

Solution of the equation (1):

1. The noise probability density is normalized:

   \[ \tilde{\mu}(z) = \sigma \tilde{\mu}(\sigma z) \]

2. The equation is solved for the normalized threshold \( \bar{\chi}_m \)

   \[ 2(K_m + 1) \int_{\bar{\chi}_m}^{\infty} (z - \bar{\chi}_m)^2 \tilde{\mu}(z) dz = \bar{\chi}_m^2 + 1 \quad (2) \]

3. We find the threshold as: \( \chi_m = \sigma \bar{\chi}_m \)
Minimax method

In the Gaussian case, the equation (2) becomes:

\[
\frac{1}{2} \text{erf} \left( \frac{\tilde{\chi}_m}{\sqrt{2}} \right) + \frac{\tilde{\chi}_m}{\tilde{\chi}_m + 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{\tilde{\chi}_m^2}{2}} = \frac{K_m}{2(1 + K_m)}.
\]

Table of numerical solutions

Is it reasonable to suppose a Gaussian PDF for the WT coefficients?

- Yes if the original noise was Gaussian
- Otherwise, other laws can be used (different laws for different levels)
Asymptotical value of the optimal threshold

If the noise wavelet coefficients have PDF:

$$\forall z \in \mathbb{R}, \quad \mu(z) = Ce^{-h(z)}$$

where $C \in \mathbb{R}^*$ and $h$ is a even continuous function, strictly increasing on $\mathbb{R}_+$ and such that

$$\lim_{z \to \infty} z^{-\beta} h(z) = \gamma \in \mathbb{R}_+, \quad \beta \geq 1$$

and

$$\forall (z_1, z_2) \in \mathbb{R}_+^2, \quad h(z_1 + z_2) \geq h(z_1) + h(z_2).$$

When $K_m \to \infty$,

$$\chi_m \sim \chi_U = h^{-1}(\ln K_m)$$

$$\Lambda_{\chi_m} \sim \frac{\chi_U^2}{\sigma^2} + 1.$$
Universal threshold

- The previous hypotheses hold for Gaussian and generalized Gaussian distributions
- In the both cases we have $\mu(z) = Ce^{-h(z)}$
- Gaussian:
  $$\mu(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{z^2}{2\sigma^2}} \quad h(z) = \frac{z^2}{2\sigma^2}$$
  $$h^{-1}(t) = \sqrt{2\sigma^2} t$$
  $$\chi_U = \sigma \sqrt{2 \ln K_m}$$
- GG:
  $$\mu(z) = Ce^{-\gamma|z|^\beta} \quad h(z) = \gamma|z|^\beta$$
  $$\chi_U = \left( \frac{\ln K_m}{\gamma} \right)^{\frac{1}{\beta}}$$
Universal threshold

- It eases the computation of the universal threshold in the Gaussian case
  - It is easier than $\chi_m$
- The difference between $\chi_U$ and $\chi_m$ can be large if $K_m$ is not large
- Nevertheless, $\chi_U$ is called universal threshold
- Using the universal threshold is referred to as visushrink
Introduction
Denoising

Universal threshold

Minimax vs. universal threshold (Gaussian case)

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<th>$K_m$</th>
<th>$\chi_m$</th>
<th>$\Lambda_{\chi_m}$</th>
<th>$\chi_U$</th>
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Threshold optimality

In the Gaussian case, the threshold estimators are asymptotically *minimax*:

If the noise coefficients are i.i.d. $\mathcal{N}(0, \sigma^2)$ then

$$\lim_{K_m \to \infty} \inf_{\hat{s}} \sup_s \left( \frac{\mathcal{E}^2(s)}{(\sigma^2 + \mathcal{E}_o^2(s)) \Lambda_{\chi m}} \right) = 1.$$

where $\inf_{\hat{s}}$ is the infimum on the set of possible estimators of $s(t)$.

Asymptotically, threshold estimators are the best in the minimax sense.
**SURE method**

**Principles**

- Minimax method is too pessimistic
- Idea: estimate the MSE in the average case, and then minimize it
- Problem: the estimation of the MSE depending on the signal
- Stein’s lemma: allows to estimate the MSE *without bias*
SURE method

Principles

- We observe the r.v. \( Y = x + Z \)
- \( x \), deterministic, is the quantity we want to measure
- \( Z \), whose standard deviation is \( \sigma \), is the noise
- We consider an estimator \( T(Y) = Y + \gamma(Y) \)
- The risk (the MSE) is:

\[
c^2(x) = E\{(x - T(Y))^2\} \\
= E\{(x - Y - \gamma(Y))^2\} \\
= \sigma^2 - 2xE[\gamma(Y)] + 2E[Y\gamma(Y)] + E[\gamma(Y)^2]
\]

Problem: the risk depends on \( x \)!
SURE method

Stein’s lemma
We observe the r.v. $Y = x + Z$ with $x$ deterministic and $Z \sim \mathcal{N}(0, \sigma^2)$; if $\gamma$ is a continuous function, piecewise derivable and such that, for all $x \in \mathbb{R}$,

$$\lim_{|z| \to \infty} \gamma(z) \exp\left(-\frac{(z - x)^2}{2\sigma^2}\right) = 0$$

$$\mathbb{E}\{\gamma(x + Z)^2\} < \infty, \quad \mathbb{E}\{|\gamma'(x + Z)|\} < \infty$$

then

$$x\mathbb{E}\{\gamma(Y)\} = \mathbb{E}\{Y\gamma(Y)\} - \sigma^2\mathbb{E}\{\gamma'(Y)\} \quad (3)$$
SURE method

Stein’s lemma

- The conditions over $\gamma(\cdot)$ are quite general
- E.g., they hold for non-linear continuous and piecewise derivable function with polynomial or slower increase
- This means that it exists $m \in \mathbb{N}$ and $A \in \mathbb{R}_+$ such that:

$$\forall z \in \mathbb{R}, |\gamma(z)| \leq A |t|^m.$$
SURE method

Stein’s lemma

Applying the Stein’s lemma, we can write the equation (3):

\[ \epsilon^2(x) = \sigma^2 + E[\gamma(Y)^2] + 2\sigma^2E[\gamma'(Y)] \]
\[ = E[J(Y)] \]

where

\[ J(Y) = \sigma^2 + 2\sigma^2\gamma'(Y) + \gamma^2(Y) \]

This is true also if \( Y = X + Z \) with \( X \) r.v. independent from \( Z \)
SURE method

Application to wavelet coefficients

- The wavelet coefficients of the signal are r.v. with finite variance, and for a given level $j$, i.i.d.
- The noise coefficients $c^b_j[k]$, are i.i.d., $\mathcal{N}(0, \sigma^2_j)$ independent from the signal
- We use soft thresholding, which verifies the hypothesis of Stein’s lemma, with

$$\gamma(z) = \begin{cases} 
\chi & \text{if } z \leq -\chi, \\
-z & \text{if } |z| \leq \chi, \\
-\chi & \text{if } z \geq \chi 
\end{cases}$$
SURE method

Application to wavelets

Estimation of the MSE: \( \hat{\epsilon}_j^2(x) = E[J(c'_j[k])] \) with:

\[
J(z) = \sigma_j^2 + 2\sigma_j^2 \gamma'(z) + \gamma^2(z)
\]

\[
= \begin{cases} 
  z^2 - \sigma_j^2 & \text{if } |z| \leq \chi \\
  \chi^2 + \sigma_j^2 & \text{if } |z| > \chi 
\end{cases}
\]

and finally:

\[
\hat{\epsilon}_j^2(x) = \frac{1}{2^{-jK}} \sum_{k=0}^{K2^{-j}-1} J(c'_j[k])
\]

We only have to find the threshold \( \chi \) minimizing \( \hat{\epsilon}_j^2 \)
**SURE method**

**Algorithm**

We sort the wavelet coefficients:

\[
A = |c^f_0| \geq |c^f_1| \geq \ldots \geq |c^f_{K2^{-j} - 1}| = B
\]

and We consider the three cases: \( \chi > A \), \( A \geq \chi \geq B \), and \( \chi < B \).

- In the first, the MSE does not depend on \( \chi \).
- In the second, \( \exists k_0 \) such that \( |c^f_{k_0}| \leq \chi < |c^f_{k_0 - 1}| \), then:

\[
2^{-j}K\hat{\epsilon}^2_j = k_0\chi^2 + (2k_0 - K2^{-j})\sigma^2 + \sum_{k=k_0}^{K2^{-j} - 1} (c^f_k)^2
\]

and the minimum is attained for \( \chi = |c^f_{k_0}| \).
- In the third, \( \hat{\epsilon}^2_j = \chi^2 + \sigma^2 \) and the minimum is for \( \chi = 0 \).
SURE method

Algorithm

- In conclusion the optimal value of $\chi$ is among:
  \[
  \{|c_j^f[0]|, |c_j^f[1]|, \ldots, |c_j^f[K2^{-j} - 1]|, 0\}
  \]
- An exhaustive search can be carried off.
- The risk can be computed with a recurrent equation.
- Total complexity: $O(2^{-j}K)$ for the search and $O(2^{-j}K \log(2^{-j}K))$ for sorting.
- Advantage: threshold automatically adapted to data.
**SURE method**

Hybrid threshold

- If the signal power, at a given resolution level is too small with respect to the noise, the SURE estimator is not reliable
- Then we use for that level the universal threshold
- Estimator of the signal power:

\[
(c_j^s)^2 = \frac{1}{K 2^{-j}} \sum_{k=0}^{K 2^{-j} - 1} (c_j^r[k])^2 - \sigma_j^2
\]

- Critical power level:

\[
\lambda_{j,K} = \frac{\sigma_j^2}{\sqrt{(K 2^{-j})}} \left[ \ln(K 2^{-j}) \right]^{3/2}
\]
SURE method

Hybrid threshold

In conclusion we use the hybrid threshold:

\[
\chi_{j,H} = \begin{cases} 
\chi_{j,\text{SURE}} & \text{if } (c_j^s)^2 > \lambda_{j,K} \\
\chi_{j,\text{U}} & \text{otherwise}
\end{cases}
\]

the universal threshold is:

\[
\chi_{j,\text{U}} = \sigma_j \sqrt{2 \ln K 2^{-j}}
\]
Example of denoising/debruit

Original signal (a), noisy, SNR = 18.86 dB (b), After universal threshold, SNR = 23.80 dB (c), denoised with sureshrink, SNR = 27.45 dB (d).
Bayes method

- We observe $Y = X + Z$, $X$ r.v. with probability density $p_X$ and $Z$ r.v. with probability density $\mu$, independent from $X$.
- Having an observation $y$ of the r.v. $Y$, we select the most probable value of $X$:

$$\hat{x} = \arg \max_x p_X(x|Y = y)$$

It is the **MAP** estimator (Maximum A Posteriori probability)
Bayes method

- MAP Estimator

\[ \hat{x} = \arg \max_{x} p_x(x | Y = y) \]

- Using Bayes rules:

\[
p_x(x | Y = y) = \frac{p_y(y | X = x)p_x(x)}{p_y(y)} = \frac{\mu(y - x)p_x(x)}{p_y(y)}
\]

- The MAP Estimator is then equivalent to:

\[ \hat{x} = \arg \min_{x} \left[ - \ln (\mu(y - x)) - \ln (p_x(x)) \right] \]

Most of reasonable of the *a priori* give thresholding estimators
Bayes method

Hypotheses

- Noise and signal mutually independent
- Their wavelet coefficients are independent r.v.
- At resolution level \( j \), the noise \( \mathcal{N}(0, \sigma_j^2) \) and the signal coefficients are i.i.d. with pdf \( p_j \)
- If the wavelet basis fits well the signal, we expect \( c_j^s[k] \) very small with high probability, and large with low probability
**Bayes method**

Laplacian PDF

\[ p_j(u) = \frac{1}{\sqrt{2\eta_j}} \exp\left(-\frac{\sqrt{2}|u|}{\eta_j}\right) \]

It can be shown that

The MAP estimator corresponding to a zero-mean Laplacian PDF with standard deviation \( \eta_j > 0 \) and with noise \( \mathcal{N}(0, \sigma_j^2) \) is the soft thresholding with threshold \( \chi_{j,B} = \sqrt{2}\sigma_j^2/\eta_j \).
Bayes method

Generalized Gaussian (GG) PDF
\[ \mathcal{G}(\alpha_j, \beta_j), (\alpha_j, \beta_j) \in (\mathbb{R}_+^*)^2, \] with:
\[ p_j(u) = \frac{\beta_j}{2\alpha_j \Gamma(1/\beta_j)} \exp\left(-\frac{|u|^{\beta_j}}{\alpha_j^{\beta_j}}\right) \]

where \( \Gamma \) the gamma function.
If \( \beta_j \leq 1 \), the MAP estimator for \( \mathcal{G}(\alpha_j, \beta_j) \) and noise \( \mathcal{N}(0, \sigma_j^2) \) is a threshold estimator with
\[ c^S_j[k] = 0 \iff |c^R_j[k]| \leq \chi_{j,B} \]

\[ \chi_{j,B} = \frac{2 - \beta_j}{2(1 - \beta_j)} \left( \frac{2\sigma_j^2(1 - \beta_j)}{\alpha_j^{\beta_j}} \right)^{1/(2 - \beta_j)}. \]
Bayes method

Generalized Gaussian (GG) PDF

When

\[ |c_j'[k]| \geq \chi_{j,B} \]

where \( \beta_j > 1 \)

the estimation of a coefficient is a \textit{shrinkage} of the observed value.

If \( \beta_j < 1 \), the estimator is rather close to the hard thresholding, since it is not continuous near the threshold.
Bayes method

Bernoulli-Gaussian PDF

$q_j[k]$: hidden random variables binary, independent and such that each component $c_j^s[k]$ of $s(t)$ is:

- carrying information, if $q_j[k] = 1$: $P(q_j[k] = 1) = \epsilon_j$
- zero, if $q_j[k] = 0$

When $q_j[k] = 1$, we suppose that $c_j^s[k]$ is Gaussian, zero-mean, with variance $\sigma_j^2$
Bayes method

Bernoulli-Gaussian PDF

Signal estimation

*Maximum A Posteriori* Estimator of \(q_j[k]\):

\[
\hat{q}_j[k] = \begin{cases} 
1 & \text{if } |c'_j[k]| > \chi_{j,B}, \\
0 & \text{otherwise}
\end{cases}
\]

threshold \(\chi_{j,B} \geq 0\):

- depends on \(\sigma^2, \sigma_j^2\) and \(\epsilon_j\)
- independent from the signal length

\[
c_j^{\hat{S}}[k] = \begin{cases} 
\frac{\sigma_j^2}{\sigma^2 + \sigma_j^2} c'_j[k] & \text{if } \hat{q}_j[k] = 1 \\
0 & \text{otherwise}
\end{cases}
\]
Bayes method

Determination of the *a priori* model parameters

The main problem is to find the parameters \((\epsilon_j \text{ and } \sigma_j^2)\)

- iterative methods (generalized likelihood method, EM algorithm, MCMC, ...)
- performance/complexity trade-off